Exam 01 (Solutions)

You will have 75-minutes to complete this exam. There are a total of 10 questions, all weighted equally. Please write legibly, clearly mark the question number, and place a box around your final answer (when there is one) clearly. Please do not write your solutions on the sheet with the exam questions. You may leave counting problems unsimplified and in terms of factorials and binomial coefficents.

1. [**WS02Q5**] (a) Write down all of the possible outcomes of flipping a coin three times. Use H for heads and T for tails. (b) Using the naïve definition of probability, what is the probability that there are never two heads in a row?

Solution: The possible options are: {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }. There are five such outcomes without two heads in a row, so $5/8 = 0.625$.

2. [**WS02Q8**] The Greek alphabet has 24 letters, of which 7 are vowels (α, ϵ, η, ι, o, ω, υ). How many two letter combinations can you create from the Greek alphabet if every combination needs at least one vowel?

Solution: There are two ways a word can be valid: either the first letter is a vowel or the second letter is a vowel. If we count these seperately and add, this will double count those words with two vowels. There are two ways of dealing with this. In both, we start by calculating how many ways a word can start with a vowel. This is equal to $7 \cdot 24$. We can now simply count those words that end in a vowel but do not start with one; this is just $(24 - 7) \cdot 7$. This gives a total of:

$$\#\{\text{num. words}\} = 7 \cdot 24 + 17 \cdot 7 = 5 \cdot (47) = 287.$$

3. [**WS03Q2**] Assume that you have a cup with m black marbles, m white marbles, and m blue marbles for some $m > 1$. Consider selecting one marble and then another marble, without putting the first one back. What is the probability that the two marbles are the same color?

Solution: The total number of possible selections is a two-stage experiment, with $3m$ options in the first round and $(3m - 1)$ options in the second round. So, there are a total of $3m \cdot (3m - 1)$ total possible ways to select two marbles. For the marbles to match, we can start with any marble we want ($3m$), but in the second round can only select the remaining $m-1$ marbles of the same color. So the probability is:

$$\frac{3m\cdot(m-1)}{3m\cdot(3m-1)}=\frac{m-1}{3m-1}.$$

4. [**WS03Q3**] Assume that birthdays are equally likely to occur in any given month. What is the probability that 6 randomly selected people will not share any birth-month?

Solution: As in the original question, we have:

$$\begin{aligned}\mathbb{P}(\text{no match}) &= \frac{(12)!/(6)!}{12^6}\\ &= 0.223\end{aligned}$$

5. [**WS03Q6**] Five students get on a bus to downtown at the same time. The bus makes three stops once it arrives in downtown Richmond. If each student randomly decides which stop to disembark, what is the probability that everyone gets off at the same stop?

Solution: Using our naïve counting definition we get:

$$\begin{aligned}\mathbb{P}(event) &= \frac{\#\{\text{ways get off same stop}\}}{\#\{\text{ways of getting off bus}\}}\\ &= \frac{3}{3^5}\\ &\approx 0.0123.\end{aligned}$$

6. [**WS04Q2**] Consider a set of cards with 4 suits/colors and 5 cards in each suit. What is the probability that a set of 5 randomly dealt cards will contain three cards of the same number and another set of two cards that share the same number?

Solution: The basic rule of counting will come into play in this example because we will take the view that putting together a full house is a four-stage experiment: pick a value you will be taking three cards from, take three cards of this value, pick a value you will be taking two cards of, take two cards of this value. Notice that the two values that are picked play different roles; if you pick a king in stage one and a four in stage three, you get a different full house than if the order was reversed. The denominator comes straight from the logic in the previous question.

Writing everything in terms of binomial coefficents, we have:

$$\begin{aligned}\mathbb{P}(\text{full house}) &= \frac{\binom{5}{1}\cdot\binom{4}{3}\cdot\binom{4}{1}\cdot\binom{4}{2}}{\binom{5\cdot 4}{5}}\\ &\approx 0.031.\end{aligned}$$

7. [**WS04Q4**] Suppose that two evenly matched teams (say team A and team B) play each other in a best-of-five games series. That is, they play up to five games, with the first team to win three games winning the series. What is the probability that A will win in exactly 4 games?

Solution:

How many ways can A win in exactly four games? To so, they have to win the fourth game and exactly 2 of the previous 3. That's just $\binom{3}{2}$. So:

$$\mathbb{P}(\text{five games}) = \frac{\binom{3}{2}}{2^4} = 0.1875.$$

There are not a lot of options and you could also brute-force this by writing down all of the options.

8. [**WS04Q7**] Consider a bag with N colored marbles inside of it where K marbles are black and $N - K$ marbles are white. You select n marbles from the bag without replacing them. What the probability that you select exactly k black marbles?

Solution: Assuming that $k \leq K$, there are $\binom{K}{k}$ options for picking the set of black balls and $\binom{N-K}{n-k}$ options for picking the white ones. The total set of choices is $\binom{N}{n}$. So, the probability is:

$$\mathbb{P}(\text{exactly k black balls}) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}.$$

9. [**WS05Q7**] How many ways are there to permute the letters in the word STATISTICS?

Solution: This is more difficult because we have three S's, three T's, and two I's. Flipping one S for another does not change the combination. We can view this as trying to put each of the letters into ten slots. Going in the order of S, T, I, A, C, we have:

$$\binom{10}{3} \cdot \binom{7}{3} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = 50400.$$

10. [**WS05Q8b**] There are 22 faculty members in the mathematics department and 8 faculty members in the data science department. We need to form a committee of 8 faculty members. How many possible committees are there if we know that there are exactly 3 mathematics faculty on the committee?

Solution:

We have a multistage experiment in which we pick the 3 mathematics faculty followed by the 5 data science faculty members. There are, by the basic rule of counting, $\binom{22}{3} \cdot \binom{8}{5}$ (or 86240) options.