Exam 05 (Solutions)

You will have 75-minutes to complete this exam. There are a total of 10 questions, each worth 10 points. Questions that include the instruction **justify** require showing your work. You need to give enough steps that show you understand the logic of the answer. You may write your answers on this sheet. Unless otherwise noted, you may start with any results from the handout.

1. [**19-01**] Let $Z \sim N(0,1)$. It can be shown that $\mathbb{E}|Z| = \sqrt{2/\pi}$. Use Markov's inequality to bound the probability $\mathbb{P}[|Z| > 3.89]$.

Solution: This is quite straightforward:

$$\mathbb{P}[|Z| \geq 3.89] \leq \frac{\sqrt{2/\pi}}{3.89} \approx 0.205$$

So, these are decreasing much (much) slower than the exact values.

2. [**19-03**] Let $Z \sim N(0,1)$. Use Chebychev's inequality to bound the probability $\mathbb{P}[|Z| > 3.89]$.

Solution: This is quite straightforward as well:

$$\mathbb{P}[|Z| \geq 3.89] \leq \frac{1}{3.89^2} \approx 0.066$$

These are tighter bounds than Markov gives, particularly for the last two.

3. [**19-06**] Chernoff's inequality has an extra term in it, the t, that provides a whole family of bounds for a given value of a. The tightest bound depends on the distribution. Let $Z \sim N(0,1)$. Using the moment generating function, what value of t provides the tightest bound on $\mathbb{E}[Z \geq a]$?

Solution: Plugging in the moment generating function, we have the following bound of Z:

$$\begin{aligned}\mathbb{P}[X > a] &\leq \frac{\mathbb{E}e^{tX}}{e^{ta}} \\ &\leq \frac{e^{\frac{1}{2}t^2}}{e^{ta}} = e^{\frac{1}{2}t^2 - ta}\end{aligned}$$

The quantity on the right will be minimized by the value in the exponent $(\frac{1}{2}t^2 - ta)$ is minimized. This is a quadratic polynomial; taking the derivative and setting it equal to 0 gives $t = a$ at the minimum, which has the following final bound on the tail probability from Chernoff's inequality for a standard normal Z:

$$\mathbb{P}[X > a] \leq e^{-\frac{1}{2}a^2}.$$

That turns out to be the correct limiting distribution of the tail of a normal.

4. [**18-01**] Let $U \sim U(0,1)$ and define $Y = U^3$. Use the change of variables formula to derive and **justify** the density of Y.

Solution: We start by determining the derivative

$$\begin{aligned}\left|\frac{d}{dy}g^{-1}(y)\right| &= \left|\frac{d}{dy}(y^{1/3})\right| \\ &= \left|\frac{1}{3}y^{-2/3}\right|\end{aligned}$$

And then:

$$\begin{aligned}f_Y(y) &= f_U(g^{-1}(y)) \cdot \left|\frac{1}{3}y^{-2/3}\right| \\ &= \frac{1}{3}y^{-2/3}\end{aligned}$$

This is just the Beta distribution with $\alpha = 1/3$ and $\beta = 1$.

5. [**18-03**] Let $Z \sim N(0,1)$ and consider the random variable $Y \sim Z^2$. We cannot directly apply the change of variables formula because $g(z) = z^2$ is not a one-to-one function (it maps positive numbers to the same number as a negative number). We can fix this by considering a random variable $X = |Z|$ and then defining Y to (equivalently) be equal to X^2. The density of X is just twice the density of a standard normal, but only for positive values of x:

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}}e^{-x^2/2}, \quad x > 0.$$

Use the change of variables formula to derive and **justify** the density of Y. You do not need to simplify the final form.

Solution: The function g is the same as in question 1, so we have:

$$\begin{aligned}\left|\frac{d}{dy}g^{-1}(y)\right| &= \left|\frac{d}{dy}\sqrt{y}\right| \\ &= \left|\frac{1}{2}y^{-1/2}\right|\end{aligned}$$

Note that this will always be positive for $y > 0$. Then, the density is given by:

$$f_Y(y) = \frac{\sqrt{2}}{\sqrt{pi}}e^{-y/2} \cdot \left[\frac{1}{2}y^{-1/2}\right], \quad y > 0$$

We will simplify this in the next question.

6. [**18-06**] Let $U \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$ be a random variable. Define $T = tan(U)$. Use the change of variable formula to determine and **justify** the form of the pdf of T.[1]

[1] The derivative of $tan^{-1}(t)$ is $1/(1+t^2)$.

Solution: Let's try to find the PDF of T. The derivative is given by:

$$\frac{du}{dt} = \frac{d}{dt}\left[tan^{-1}(t)\right] = \frac{1}{t^2+1}$$

Note that the density of U is $\frac{1}{\pi}$ for all values of u between $\pm\pi/2$. This gives then that:

$$\begin{aligned} f_T(y) &= f_U(u) \cdot |det(J_{g^{-1}})| \\ &= \frac{1}{\pi} \cdot \frac{1}{t^2+1} \\ &= \frac{1}{\pi(t^2+1)} \end{aligned}$$

This example presents an interesting justification of the Cauchy distribution. Stand in front of a designated point on an infinitely long wall holding a flashlight. Pick an angle uniformily somewhere between your immediate left and your immediate right and shine the flashlight at the wall from this angle. The location of the light on the wall is given by the tangent of the angle you choose relative to the wall. Therefore, the location on the wall will follow a Cauchy distribution. While this setup may sound quite strange, it is a great mental image to help understand some of the seemingly odd behaviors of the distribution.

7. [**20-1**] Let $X_1, \ldots, X_n$ be a sequence of n i.i.d. continous random variables that have a pdf $f(x)$ and a cdf $F(x)$. Define the random variables $Y_1, \ldots, Y_n$ be the corresponding order statistics, with density function $g_j(y)$ and $G_j(y)$, respectively. **Justify** the form of $G_n(y)$—the cdf of the **maximum** value—in terms of n and F.

Solution: In order for the maximum to be less than y, all values of X_j must be less than y. So, we have:

$$\begin{aligned} G_n(y) &= \mathbb{P}[Y_n \leq y] \\ &= \prod_j \mathbb{P}[X_j \leq y] \\ &= \prod_j F(y) \\ &= [F(y)]^n \end{aligned}$$

So, it's just the cdf of X_j raised to the power of n.

8. [**20-2**] Let $X_1, \ldots, X_n$ be a sequence of n i.i.d. continous random variables that have a pdf $f(x)$ and a cdf $F(x)$. Fix a value y and a positive value Δ. What is the joint probability that $X_1, \ldots, X_{k-1}$ are all less than y, that $X_{k+1}, \ldots, X_n$ are all greater than $y + \Delta$, and that X_k is in the interval $[y, y + \Delta]$?

Solution: Here are the three different components:

$$\begin{aligned}
\mathbb{P}[X_1 \le y] \times \cdots \times \mathbb{P}[X_{k-1} \le y] &= [F(y)]^{k-1} \\
\mathbb{P}[X_{k+1} \ge y + \Delta] \times \cdots \times \mathbb{P}[X_n \ge y + \Delta] &= [1 - F(y)]^{n-k} \\
\mathbb{P}[X_k \in [y, y + \Delta]] &= F(y + \Delta) - F(y)
\end{aligned}$$

Multiplying these together, we have:

$$[F(y)]^{k-1} \times [1 - F(y + \Delta)]^{n-k} \times [F(y + \Delta) - F(y)]$$

9. [**20-7**] The density of the kth order statistic from a set of n i.i.d. random variables with a cdf of $F(y)$ and a pdf of $f(y)$ is given by:

$$g_k(y) = \left[\frac{n!}{(n-k)!(k-1)!}\right] \times [F(y)]^{k-1} \times [1 - F(y)]^{n-k} \times f(y)$$

Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} U(0, 1)$. **Justify** that the kth order statistic has a Beta distribution while deriving the values of α and β by plugging in the values of f, F, and converting the factorials using the gamma function.[2]

[2] Hint: $\Gamma(n) = (n-1)!$

Solution: We have:

$$\begin{aligned}
g_k(y) &= y^{k-1} \cdot (1-y)^{n-k} \times \left[n \cdot \binom{n-1}{k-1}\right] \\
&= y^{k-1} \cdot (1-y)^{n-k} \times \left[n \cdot \frac{(n-1)!}{(k-1)!(n-k)!}\right] \\
&= y^{k-1} \cdot (1-y)^{n-k} \times \left[\frac{(n)!}{(k-1)!(n-k)!}\right] \\
&= y^{k-1} \cdot (1-y)^{n-k} \times \left[\frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}\right]
\end{aligned}$$

Setting $\alpha = k$ and $\beta = n - k + 1$, we get:

$$\begin{aligned}
g_k(y) &= y^{k-1} \cdot (1-y)^{n-k} \times \left[\frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}\right] \\
&= y^{\alpha-1} \cdot (1-y)^{\beta-1} \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]
\end{aligned}$$

And this is the density of the Beta distribution with parameters α and β. So, we finally see a justification for the form (and have a full derivation of the normalizing constant) of a Beta distribution.

10. [**18-7**] What is an adjective describing how happy you are that there is no final exam for this class.

Solution: Thesaurus.com suggests the following possibilities: cheerful, contented, delighted, ecstatic, elated, glad, joyful, joyous, jubilant, lively, merry, overjoyed, peaceful, pleasant, pleased, satisfied, thrilled, upbeat, blessed, blest, blissful, blithe, can't complain, captivated, chipper, chirpy, content, convivial, exultant, flying high, gay, gleeful, gratified, intoxicated, jolly, laughing, light, looking good, mirthful , on cloud nine, peppy, perky, playful, sparkling, sunny, tickled, tickled pink, up, walking on air. Other answers are also possible.