Handout 03: Ordered sampling with(out) replacement

Today we continue our study of counting, which we saw in the previous handout is closely tied to many concepts in probability theory. We will focus on two results that formalize patterns you saw on questions from the previous worksheet.

Theorem 3.1 (Ordered sampling with replacement) *Consider selecting k objects from a collection of n objects with replacement, so that choosing a certain object does not stop us from selecting it again. Then there are n^k possible ordered outcomes.*

Theorem 3.2 (Ordered sampling without replacement) *Consider selecting k objects from a collection of n objects without replacement, so that any object can be selected at most once. The number of ordered outcomes if $k \leq n$ is given by:*

$$n \cdot (n-1) \cdots (n-(k-1)) = \frac{n!}{(n-k)!}.$$

When $k > n$ there are no possible outcomes.

Notice that in both of these definitions, we care about the order in which the results were obtained. Consider the special case of sampling n elements without replacement from a set of size n. We always know the elements we will select (all of them!); it is only the order that is important. We can view this as just arranging a set of n elements in order. For example, the six ways of arranging the numbers 1, 2, and 3 are:

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), \text{ and } (3,2,1).$$

The ways or arranging a set of objects is sufficently important that we will give them their own name and formula.

Theorem 3.3 (Permutation of a set) *A* permutation *of a collection of n objects is a fixed ordering of the objects. There are a total of $n!$ permutations of n objects.*

Proof. As described above, a permutation can be viewed as sampling n elements from a set of n elements without replacement. Applying Theorem 3.2 we see that there are $\frac{n!}{(n-n)!} = n!$ such outcomes ■.