Handout 04: Unordered sampling and partitions

A standard technique when counting is to consider counting a larger thing, in which every element is over counted a certain number of times. The desired result comes from dividing the larger count by the number of times each thing is over counted. This technique will help extend the results from last time to unordered sampling.

Theorem 4.1 (Unordered sampling without replacement) *Consider selecting k objects from a collection of n objects without replacement, so that any object can be selected at most once. The total number of unordered outcomes if $k \leq n$ is given by the following notation and formula:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$, then there are no such outcomes.

Proof. From Theorem 3.2, there are $\frac{n!}{(n-k)!}$ total ways of selecting k elements from n elements without replacement if we consider different orderings of the elements to be distinct. This overcounts the result we want by the number of ways that the k selected elements can be permuted, which from Theorem 3.3 is given by $k!$. Dividing the number of ordered collections by the amount of overcounting yields the result. ■.

The notation in the previous theorem is called a binomial coefficent, and is pronounced "n choose k." This object turns up all over combinatorics and probability theory and is a helpful shorthand for writing equations in terms of factorials. To finish our counting of sampling k from a set of size n, let's count the number of unordered sets with replacement. We will see that this can be written in terms of the binomial coefficent. A proof will be derived on the worksheet.

Theorem 4.2 (Unordered sampling with replacement) *Consider selecting k objects from a collection of n objects with replacement. The total number of unordered outcomes is given by:*

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

Now, we will end out unit on counting with a slightly different type of result. Let's consider enumerating how many different ways a population of n items can be partitioned into groups of different sizes.

Theorem 4.3 (Partitions) *Let* n *be a positive integer and* $k_1, k_2, \ldots,$ k_r *be nonnegative integers such that* $\sum_{i=1}^{r} k_i$ *is equal to* n. *The number* *of partitions of* n *objects into groups of sizes* k_1, k_2,..., k_r *is given by:*

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! \cdot k_2! \cdots k_r!}$$

The results we have dervied on the past few notes will get us a long way in our study of discrete probability. There are many other counting theorems, though most are less obviously applicable in probability theory. If you are interested, I suggest you find a good combinatorics book (or better yet, course), which extends on the basic ideas explored here.