Handout 06: Set Theory and Probability

A set is an unordered collection of unique elements. Typically, we will talk about sets having elements that all come from some larger containing set called a **sample space**, which we will denote as S. Given arbitrary sets A and B from a sample space S, we have the following definitions and notation.

- We write $a \in A$ to indicate that a is an **element** of the set A.
- A is a **subset** of B, $A \subseteq B$, if all elements of A are also in B.
- The **union** $A \cup B$ consists of every element in either A or B.
- The **intersection** $A \cap B$ consists of elements in both A and B.
- The **difference** $A - B$ are elements that are in A but not in B.
- The **complement** of A, written A^c, is $S - A$, everything not in A.
- The **cardinality** of A, written $|A|$, measures the size A.
- The **empty set**, written $\emptyset$, is a set with no elements.

There are a variety of different theorems that we could prove describing how these different operations relate to one another. I think it is better to avoid dumping them all on you at once and derive them as we need them. We start with a few of the most useful results on today's worksheet.

We will use set notation to give a non-naïve definition of probability. Our sample space S will indicate all of the possible unique outcomes of some random event. We will use the term **event** for sets A that are subsets S. Finally, we can define a probability function $\mathbb{P}$ that maps each event to a probability value between 0 and 1. The probability function must have certain properties that match our conventional understanding of probability, as described in the following definition.

Definition 6.1 *A probability function $\mathbb{P}$ for a sample space S associates a probability between 0 and 1 to subsets of S such that:*

1. $\mathbb{P}[S] = 1$.
2. *For every pair of events A and B such that $A \cap B = \emptyset$ we have:*[1]

$$\mathbb{P}\left(A \cup B\right) = \mathbb{P}A + \mathbb{P}B.$$

The pair $(S, \mathbb{P})$ is called a probability space.

[1] A pair of events such as these that have no common elements are called *mutually exclusive.*

From this definition, the naïve definition of probability follows from a particular choice of a probability function:

$$\mathbb{P}_{\text{naïve}}\left(A\right) = \frac{|A|}{|S|}.$$

There are many important consequences that derive from the definition of a probability function. We will explore these over the next few class meetings.