Handout 10: Random Variables

A **random variable** is a function that maps a sample space into the real line.[1] We can describe a random variable through the cumulative distribution function (cdf), given by:

$$F_X(x) = \mathbb{P}[X \leq x], \quad x \in \mathbb{R}.$$

We can also represent it by the probability mass function (pmf), given by:[2]

$$p_X(x) = \mathbb{P}[X = x].$$

The pmf is usually more intuitive and easier to derive. The cdf has the advantage that it can be more easily extended to continuous distributions.

[1] We often use a capital letter, frequently X or Y, rather than standard functional notation to denote a random variable.

[2] For now we will assume that the random of random variable is at most a countably infinite set of values, called a discrete random variable.

Two random variables X and Y are **independent** if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all values of x and y. We can create new random variables by defining arithmetic transformations of existing random variables. For example, $Z = X^2$ or $Z = X + Y$.

The **expected value** of a random variable is defined and denoted by:[3]

$$\mathbb{E}[X] = \sum_x x \cdot p_X(x) = \sum_x x \cdot \mathbb{P}[x = X],$$

Where the sum is taken over all values where $p_X(x)$ is non-zero. For any two random variables X and Y and contant $\alpha \in \mathbb{R}$, we have:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$
$$\mathbb{E}[\alpha \cdot X] = \alpha \cdot \mathbb{E}[X]$$

If X and Y are independent, we also have:

$$\mathbb{E}[X \times Y] = \mathbb{E}[X] \times \mathbb{E}[Y]$$

These results come directly from the definition of the expected value.

[3] If we run an experiment a large number of times, we will expect that the average value of an observed random variable will converge to its expected value.