Handout 19: Tail Bounds

A tail bound is a commonly used tool in probability theory that provides a way of measuring the probability that a random variable is a certain distance away from the mean (or origin). When the distribution of a random variable is known, we can usually compute an exact tail bound for any given probability or distance using computational methods to approximate the cdf. When the distribution is unknown, there are several methods for computing upper bounds on the tail probabilities. Let's see examples of both of these.

Let $Z \sim N(0, 1)$, a standard normal. Using a numeric approximation to the cdf (which we have called Φ), we can derive the following well-known bounds on the probability that Z will be too far away from zero:

$$\begin{aligned}
\mathbb{P}[|Z| > 1.28] &\approx 0.20 \\
\mathbb{P}[|Z| > 1.96] &\approx 0.05 \\
\mathbb{P}[|Z| > 2.58] &\approx 0.01 \\
\mathbb{P}[|Z| > 3.89] &\approx 0.0001.
\end{aligned}$$

These are particularly useful because, as we have shown, any normally distributed random variable $X \sim N(\mu, \sigma^2)$ can be written as $X = \mu + \sigma Z$, making these bounds applicable to any normal distribution. Further, many distributions can be approximated by the normal with the CLT.

Now, to a more general result. Let X be a random variable and a be a positive constant. Then, **Markov's Inequality** gives that:

$$\mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}|X|}{a}.$$

The proof of this is relatively straightforward. Let $Y = |X|/a$. Then, let V be a random variable that is 0 if $Y < 0$ and 1 if $Y > 1$. It should be clear that $V \leq Y$. Taking the expected value of both sides gives $\mathbb{P}[Y > 1] \leq \mathbb{E}Y$. Plugging in the equation for Y completes the result. The two other general tail bound inequalities on the reference sheet will be derived by applying Markov's Inequality to the questions on the worksheet.