Worksheet 02 (Solutions)

1. Working in pairs (or triples if needed), select three marbles of one color and three marbles of another color. Pick one color to a reference color, which we will call C, and put all the marbles in a cup. Using the naïve definition of probability, what is the probability that a randomly selected marbel from the cup will be the color C?[1]

[1] I know the answer likely seems obvious, but try to actually write down the steps of the naïve definition.

Solution: There are six total outcomes (each of the marbles), three of which are the desired outcome of selecting the color C. So:

$$\mathbb{P}\left(\text{color C}\right) = \frac{\#\{3\}}{\#\{6\}} = 0.5$$

2. The concept of the "probability of an event" can be defined as the proportion of times we would expect the event to occur as the outcome of a random "experiment" if the experiment is repeated a sufficently large number of times. We can use a simulation to approximate a probability using the empirical probability, defined as:

$$\text{empirical probability}\left(\text{event}\right) = \frac{\#\{\text{times event occured}\}}{\#\{\text{total trials}\}}$$

Note that this closely resembles the naïve definition of probability. Simulate the probability of choosing the color C from the cup with 12 random draws. How closely does your empirical probability match the one your calculated before?

Solution: The specific results will vary.

3. Consider selecting one marble and then another from the cup, without replacing the first marble when selecting the second. Using the naïve definition of probability and the basic rule of counting, what is the probability that both marbles are the same color?[2]

[2] Do not resort to counting all the possibilites. Try to use the basic rule of counting.

Solution: This is slightly more difficult than it might at first appear. The total number of events is straightforward: the first stage has 6 possible marbles and the second has 5, so in total there are $6 \cdot 5 = 30$ possible outcomes. For the event of matching color, we can start with any marbles we want, so there are 6 possible options. But for the second marble, we have to match the first, and there will be only 2 remaining marbles with the same color (since we already have one). So:

$$\mathbb{P}\left(\text{matching colors}\right) = \frac{6 \cdot 2}{6 \cdot 5} = \frac{2}{5} = 0.4$$

It should be intuitive that this is slightly less than one half.

4. Simulate the previous question with 12 random draws. Compare to the analytic solution.

Solution: The specific results will vary.

5. Write down all of the possible outcomes of flipping a coin three times. Use H for heads and T for tails. What is the probability that all of the flips have the same result (in other words, 3 heads or 3 tails)?

Solution: The possible options are: {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }. There are four such outcomes. The probability that all are the same is given by: $2/8 = 1/4 = 0.25$.

6. Consider a set of cards with 3 colors/suits and 10 cards of each suit. If we select 5 cards from the deck, what's the probability that all of them are the same color/suit? This called a *flush* in poker. Note: Try to compute the numerical answer with a calculator.

Solution: Let's count how many hands of cards there are that all have the same suit. While this problem does not care about the ordering of the cards, we can consider the card order as long as we count it both in the numerator and denominator.

There are $3 \cdot 10 = 30$ possible outcomes for the first card, as every card could potentially be part of a flush. Once we have the first card, the second card has to be of the same suit. As we already have one card from this suit there are $10 - 1 = 9$ possible options for the second card that preserves the flush. Similarly, there are 8 possible third cards, 7 possible fourth cards, and 6 possible fifth cards. So, there are $30 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ ways of being dealt a flush.

How many total ways are there to be dealt cards? This is just $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$. Therefore, the probability of a flush is simply:

$$\begin{aligned}\mathbb{P}\,(\text{flush}) &= \frac{30 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6}{29 \cdot 28 \cdot 27 \cdot 26}\end{aligned}$$

If we plug this in, the result is 0.00530504, so only about one set of cards in every two hundred that is dealt will all be the same suit.

7. Re-write your previous solution using only factorials.

Solution: We can re-write this as follows:

$$\begin{aligned}\mathbb{P}\left(\text{flush}\right) &= \frac{9\cdot 8\cdot 7\cdot 6}{29\cdot 28\cdot 27\cdot 26}\\ &= \frac{9!}{5!}\cdot\frac{25!}{29!}.\end{aligned}$$

8. Using the 26 letters in the latin alphabet, how many two-letter 'words' can you construct? These do not need to be actual words; just count the unique combinations.

Solution: Using the basic rule of counting, this is simply $26\cdot 26 = 26^2$.

9. How many combinations are there if every word needs a vowel (a, e, i, o, u)? How many combinations are there if we also allow words that end in y (like 'my')?

Solution: FIRST PART: There are two ways a word can be valid: either the first letter is a vowel or the second letter is a vowel. If we count these seperately and add, this will double count those words with two vowels. There are two ways of dealing with this. In both, we start by calculating how many ways a word can start with a vowel. This is equal to $5\cdot 26$. We can now simply count those words that end in a vowel but do not start with one; this is just $21\cdot 5$. This gives a total of:

$$\#\{\text{num. words}\} = 5\cdot 26 + 21\cdot 5 = 5\cdot(47) = 235.$$

Similarly, we can double count those words that have two vowels and then subtract them off at the end:

$$\#\{\text{num. words}\} = 5\cdot 26 + 5\cdot 26 - 5\cdot 5 = 5\cdot(47) = 235.$$

These, thankfully, yield the same result.

SECOND PART: For the second part, we can again do one of two ways. If we count all of the words with a vowel in the first spot and then add all of the words that end in a vowel or 'y' but do not start with a vowel, we get:

$$\#\{\text{num. words}\} = 5\cdot 26 + 21\cdot 6 = 256.$$

Or, we can count all of the words that are valid based on the first letter, add those valid based on the second letter, and then substract the number of words valid based on the first and the second letter:

$$\#\{\text{num. words}\} = 5\cdot 26 + 26\cdot 6 - 5\cdot 6 = 256.$$

Again, getting the same answer.

10. How many 'words' are there if we have three letters, without any vowel restrictions? How about four letters? What about words of length k?

Solution: Three letter words will have 26^3 and four letter words will have 26^4 options. In generaly, with k length words, we have 26^k options.

11. $(\star)$[3] Consider a set of cards with 3 colors/suits and 10 cards of each suit. If we select $k+1$ cards from the deck for k between 1 and 9, write down the probability that all of the k cards have the same suit using factorials. Compute the probability for $k = 1, 2, 3, 4, 5, 6$.

[3] A star indicates that this will not appear as a normal exam question. It is usually for particularly involved questions that have a lot of parts.

Solution: Thinking of this as a multistage experiment, the number of ways of getting any specific sequence of cards is $30 \cdot 29 \cdots (30 - k)$ and the number of ways of getting a flush is $30 \cdot 9 \cdot 8 \cdots (9 - k + 1)$. We can write these as $\frac{30!}{(30-k-1)!}$ and $\frac{30 \cdot 9!}{(9-k)!}$. Writing this as a fraction and simplifying becomes:

$$
\begin{aligned}
\mathbb{P}\left(\text{flush}\right) &= \frac{\frac{30 \cdot 9!}{(9-k)!}}{\frac{30!}{(30-k-1)!}} \\
&= \frac{\frac{9!}{(9-k)!}}{\frac{29!}{(29-k)!}} \\
&= \frac{9!}{(9-k)!} \cdot \frac{(29-k)!}{29!}.
\end{aligned}
$$

Plugging in the values of k gives: 0.310, 0.089, 0.023, 0.005, 0.001, 0.00018.

The hardest part is just making sure that you have all of the indices correct. It's easy to accidentally forget a ± 1 somewhere in the equation.