Worksheet 04 (Solutions)

1. Consider a deck of cards with 5 suits and 15 cards from each suit. How many different hands of 4 cards can you be dealt if we do not care about the order of the cards that we are given?

Solution: This is just a binomial coefficent:

$$\binom{5 \cdot 15}{4} = \frac{75!}{71!4!} = 1,215,450.$$

You might have trouble typing in the right-hand side directly in a calculator, but using either log factorials or a custom binomial coefficent function should work.

2. Consider a set of cards with 4 suits/colors and 10 cards in each suit. What is the probability that a set of 5 randomly dealt cards will contain three cards of the same number and another set of two cards that share the same number? In poker this is called a *full house*.

Solution: The basic rule of counting will come into play in this example because we will take the view that putting together a full house is a four-stage experiment: pick a value you will be taking three cards from, take three cards of this value, pick a value you will be taking two cards of, take two cards of this value. Notice that the two values that are picked play different roles; if you pick a king in stage one and a four in stage three, you get a different full house than if the order was reversed. The denominator comes straight from the logic in the previous question.

Writing everything in terms of binomial coefficents, we have:

$$\begin{aligned} \mathbb{P}(\text{full house}) &= \frac{\binom{10}{1} \cdot \binom{4}{3} \cdot \binom{9}{1} \cdot \binom{4}{2}}{\binom{10 \cdot 4}{5}} \\ &\approx 0.0033. \end{aligned}$$

3. Consider a set of cards with 7 suits/colors and 10 cards in each suit. What is the probability that a set of 5 randomly dealt cards will have three cards of the same number but without having either four or five cards of the same type or three cards of one number and two matching cards of another number? That is, you have three cards of equal numbers but the other two are of numbers that are different from each other and from the set of three.

Solution: It is helpful again think of assembling the hand as a multi-stage experiment. We select the number that exists in triplicate and select three cards from this set. And then, we pick the two numbers that will happen for the remaining two cards and then select their color. So:

$$\begin{aligned}\mathbb{P}(\text{three of a kind}) &= \frac{\binom{10}{1}\cdot\binom{7}{3}\cdot\binom{9}{2}\cdot\binom{7}{1}\cdot\binom{7}{1}}{\binom{7\cdot 10}{5}}\\ &\approx 0.051.\end{aligned}$$

Given the large number of suits, the probability is actually not too small. The desired hand occurs a little more than 5% of the time.

You might notice that in the five-stage experiment it's not clear when we pick the two numbers in the $\binom{9}{2}$ part which order we will select the two cards' suits. To be more precise, assume that we will pick the small number's suit first and the larger number's suit second. That makes the process well defined while avoiding any overcounting.

4. Suppose that two evenly matched teams (say team A and team B) make it to the baseball World Series. We can model the outcome of a game as coin flip that has the letter A one side and the letter B on the other side. The series ends as soon as one of the teams has won four games. Thus, it can end as early as the 4th game (a "sweep") or as late as the 7th game, with one team winning its fourth game compared to the other team's three wins. What's the probability that team A wins the in exactly (a) 4 games, (b) 5 games, (c) 6 games, or (d) 7 games? Add the four numbers to see that the sum to 0.5 as you would (hopefully) expect.

Solution:

(a) This is just an application of the naïve definition of probability again. To end in exactly four games, the pattern of winning outcomes must be $AAAA$, so there is only one way it can happen. So:

$$\mathbb{P}(\text{four games}) = \frac{1}{2^4} = 0.0625.$$

(b) How many ways can A win in exactly five games? To so, they have to win the fifth game and exactly 3 of the previous 4. That's just $\binom{4}{3}$. So:

$$\mathbb{P}(\text{five games}) = \frac{\binom{4}{3}}{2^5} = 0.125.$$

(c) Similarly, six games requires winning the 6th game and exactly 3 of the previous 5. So:

$$\mathbb{P}(\text{six games}) = \frac{\binom{5}{3}}{2^6} = 0.156.$$

(d) Finally, seven games requires winning the 7th game and exactly 3 of the previous 6. So:

$$\mathbb{P}(\text{seven games}) = \frac{\binom{6}{3}}{2^7} = 0.156.$$

(e) Adding these gives:

$$0.0625 + 0.125 + 0.156 + 0.156 = 0.4995$$

It would be exactly 0.5 if we kept enough significant digits. Since both teams are evenly matched, we would expect team A to win the series exactly half of the time.

5. It is an easy mathematical result that $\binom{n}{k} = \binom{n}{k,n-k}$. Describe *qualitatively* why this equality must.

Solution: The quantity $\binom{n}{k}$ measures how many ways we can select k things from a set of n things without replacement. This is exactly the same as spliting a set of n things into a pile of k things and another pile of $n-k$ things if we think of one pile as the things selected and the other pile as things not selected.

6. Prove the formula on the handout for the number of partitions of n things into r groups when r is equal to 3. You should see that the general formula for an aribrary r is relatively clear once you see a specific case.

Solution: We want to split the n things into three groups of sizes k_1, k_2, and k_3. This is a multistage experiment: the first part is to select the k_1 elements from the set of n elements, for which there are $\binom{n}{k_1}$ options. In the second stage, there are $n - k_1$ elements remaining and we want to select k_2; there are $\binom{n-k_1}{k_2}$ ways to do this. In the final stage we need to select k_3 elements from the $n - k_1 - k_2$ elements. There is only one way to do this since $k_1 + k_2 + k_3 = n$, but it will actually be better to write this out as $\binom{n-k_1-k_2}{k_3}$.

Now, since this is a three-stage experiment where the number of options in each stage is fixed, the total number of partitions is just their product:

$$
\begin{aligned}
\binom{n}{k_1, k_2, k_3} &= \binom{n}{k_1} \cdot \binom{n-k_1}{k_2} \cdot \binom{n-k_1-k_2}{k_3} \\
&= \frac{n!}{k_1!(n-k_1)!} \cdot \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} \cdot \frac{(n-k_1-k_2)!}{k_3!(n-k_1-k_2-k_3)!} \\
&= \frac{n!}{k_1 \cdot k_2 \cdot k_3 \cdot (0)!} \\
&= \frac{n!}{k_1 \cdot k_2 \cdot k_3}
\end{aligned}
$$

It should be clear why this would work for a different value of r. You could do a formal proof of the general result using induction (doing it directly is a notational mess).

7. Consider a bag with N colored marbles inside of it where K marbles are black and $N-K$ marbles are white. You select n marbles from the bag without replacing them. What the probability that you select exactly k black marbles?[1]

[1] This is the density of something called the hypergeometric distribution. Come back to you solution in a few weeks after we have formalized the concept of a random variable to see the connection.

Solution: Assuming that $k \leq K$, there are $\binom{K}{k}$ options for picking the set of black balls and $\binom{N-K}{n-k}$ options for picking the white ones. The total set of choices is $\binom{N}{n}$. So, the probability is:

$$
\mathbb{P}(\text{exactly k black balls}) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}.
$$

8. ($\star$) Let's prove Theorem 4.2. I will mostly give you the ideas but want you to understand each of the steps. (a) To start, convince yourself that the number of ways to select k things from a set of n with replacement when we do not care about the order is equivalent to finding the number of non-negative integer solutions to the following equation:

$$
k = x_1 + x_2 + \cdots x_n = \sum_{i}^{n} x_i, \quad x_i \in \{0, 1, 2, \ldots\}
$$

By thinking of x_i as the number of items that were selected from the i'th element. (b) Now, the trickier bit. Consider a sequence of $(n+k-1)$ boxes. For example consider $n = 5$ and $k = 13$; here are $(n+k-1) = 17$ boxes:

□□□□□□□□□□□□□□□□□

Consider coloring in k of the boxes. Continuing with our example, here are 13 of the boxes filled in:

■■□■□■■■□■■■■■■■ □

Now, take all of adjacent black squares, count them, and turn them into their respective counts. If two unshaded boxes appear next to another, put a zero between them. Similarly, if the sequence starts with an unshaded box, put a zero at the front and if it ends with an unshaded box put a zero at the end. For example:

$$2\square 1\square 3\square 7\square 0$$

Why will this always give n numbers that sum to k? (c) Convince yourself that the number of ways of shading in k of the $(n+k-1)$ boxes is equivalent to the number of non-negative integer solutions to $k = \sum_i^n x_i$. Finally, (d) put all of the parts together to prove Theorem 4.2.

Solution: There's not much more to say for (a) and (b) as I already gave you the main details. Hopefully the example helps if you're still stuck. With (c), we will always have $n+k-1-k = n-1$ unshaded boxes. We will have non-negative integers (possibly zeros) at the start of the sequence and after each of the unshaded boxes, so a total of n numbers. The numbers come from aggregating up the k shaded boxes, plus some number of zeros, so the total count must be k.

For (d), all we need to count is the number of ways of shading in k boxes from a set of $n+k-1$ boxes. This is equivalent to selecting an unordered sample from a set without replacement, since we don't care about the order the boxes are shaded and we can only shade a box once. So, the number of ways is just $\binom{n+k-1}{k}$.