Worksheet 07 (Solutions)

1. The workers in a particular factory are 65% tall, 70% married, and 45% married and tall. If a worker is selected at random from this factory, find the probability that the worker is (a) a married non-tall person, (b) a single non-tall person, (c) married or tall or both.

Solution: Build a table!

	Tall	Non-Tall	Total
Married	0.45	0.25	0.70
Single	0.20	0.10	0.30
Total	0.65	0.35	1.00

The answers just roll right off the table: (a) 25%, (b) 10%, and (c) 90%. The last one is also just 1 minus the number of single non-tall people.

2. Psychology majors are required to take two particular courses: PSYC 100 and PSYC 200. It is known that the chances of getting an A in PSYC 100 is .4 and the chances of getting an A in PSYC 200 is .3, while the chances of getting an A in both courses are .05. What are the chances that a randomly selected student will get at least one A in the two courses?

Solution: Another table!

	PSYC 100: A	PSYC 100: Non-A	Total
PSYC 200: A	0.05	0.25	0.30
PSYC 200: Non-A	0.35	0.35	0.70
Total	0.40	0.60	1.00

So the answer is $0.05 + 0.25 + 0.35$, or 65%.

3. You give a friend a letter to mail. They forget to mail it with probability .2. Given that the letter is mailed it, the Post Office delivers it with probability .9. If the letter was not delivered, what is the probability that it was not mailed?

Solution: Another table! Let D be the event that the letter is delivered and M be the event that it was mailed. There are two differences from the first two more straightforward questions. We can reason an additional piece of information: $\mathbb{P}(D \cap M^c) = 0$. This is because a letter cannot be delivered if it is not mailed. Also, the question gives us a conditional probability ($\mathbb{P}(D|M) = 0.9$) and we need to work backwards to get the intersection:

$$\begin{aligned}
\mathbb{P}(D|M) &= \frac{\mathbb{P}(D \cap M)}{\mathbb{P}(M)} \\
0.9 &= \frac{\mathbb{P}(D \cap M)}{0.8} \\
0.9 \cdot 0.8 &= \mathbb{P}(D \cap M) \\
0.72 &= \mathbb{P}(D \cap M)
\end{aligned}$$

I used the fact that the probability of mailing the letter is 0.8 since the probability of not mailing it is 0.2. Then:

	M	M^c	Total
D	0.72	0.00	0.72
D^c	0.08	0.20	0.28
Total	0.80	0.20	1.00

Then, the probability that it was not mailed if it was not delivered is just:

$$\begin{aligned}
\mathbb{P}(M^c|D^c) &= \frac{\mathbb{P}(M^c \cap D^c)}{\mathbb{P}(D^c)} \\
&= \frac{0.2}{0.28} \\
&\approx 0.714.
\end{aligned}$$

And that's it. You could have avoided filling in a few parts of the table; However, I find it easier to do the whole thing and then answer the question.

4. A student knows the answer to 40 of 50 multiple choice questions on an exam. They select an answer at random (from among 5 possible answers) for the remaining 10 questions. What is the probability that the student actually knew the answer to a particular question that they got correct?

Solution: This is just another conditional probability question that is essentially the same as the previous question. Let K be the student knows the answer and C be the event that it is correct. We know that the probability of $K \cap C^c$ since the student does not get anything wrong that they know. Then, we have another conditional probability in the question:

$$\begin{aligned}
\mathbb{P}(C|K^c) &= \frac{\mathbb{P}(C \cap K^c)}{\mathbb{P}(K^c)} \\
0.2 &= \frac{\mathbb{P}(C \cap K^c)}{0.2} \\
0.2 \cdot 0.2 &= \mathbb{P}(C \cap K^c) \\
0.04 &= \mathbb{P}(C \cap K^c)
\end{aligned}$$

Then, the table is just:

	K	K^c	Total
C	0.80	0.04	0.84
C^c	0.00	0.16	0.16
Total	0.80	0.20	1.00

And the question we wanted to answer is:

$$\begin{aligned}
\mathbb{P}(K|C) &= \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C)} \\
&= \frac{0.8}{0.84} \\
&\approx 0.952.
\end{aligned}$$

5. We have a coin that comes up heads with probability 0.6. Assume that we flip the coin 5 times. Write H_j as the event that the j'th flip is heads. Assume that all of the events H_j are independent. What is the probability that the first 3 flips are heads and the next 2 flips are tails?

Solution: We can now do probability questions the way many of you wanted during the first few weeks. We know that $\mathbb{P}H_j = 0.6$ and $\mathbb{P}H_j^c = 0.4$. Since all the events are independent, we have:

$$\begin{aligned}
\mathbb{P}(\{HHHTT\}) &= \mathbb{P}(H_1 \cap H_2 \cap H_3 \cap H_4^c \cap H_5^c) \\
&= \mathbb{P}(H_1) \cdot \mathbb{P}(H_2) \cdot \mathbb{P}(H_3) \cdot \mathbb{P}(H_4^c) \cdot \mathbb{P}(H_5^c) \\
&= 0.6 \cdot 0.6 \cdot 0.6 \cdot 0.4 \cdot 0.4 \\
&= 0.6^3 \cdot 0.4^2 \approx 0.0345
\end{aligned}$$

6. Continuing from the last question, notice that your answer is the probability of getting any sequence of five heads and tails where exactly three of them are heads.[1]. What, then, is the probability of getting

[1] So, it would apply for example to the probability of getting HTHTH and TTTHH as well

exactly 3 heads in the sequence of 5 flips?

Solution:

The event that any specific sequence of H's and T's with 3 heads is $0.6^3 \cdot 0.4^2$. Each of these events is a unique, specific element of the overarching sample space S, so each event is mutually exclusive. The probability of the event that we want is, therefore, just this number times the number of different ways that we can have a sequence with exactly 3 heads. We've already done this: it's just $\binom{5}{3}$, by the logic that we need to select the three flips from the five possible flips in which the heads occur. So:

$$\begin{aligned}\mathbb{P}(\text{exactly 3 heads}) &= \binom{5}{3} \cdot 0.6^3 \cdot 0.4^2 \\ &=\approx 0.3456\end{aligned}$$

7. ($\star$) Consider a coin that is heads with probability p. Write down a general formula for the probability that there will be exactly k heads in a sequence of n flips using the coin.

Solution: I put a star on this because it might be a bit abstract if you are new to these ideas, but we basically just directly apply the solutions to the previous questions with the 0.6 replaced with p, the 3 replaced with k, and the 5 replaced with n.

$$\mathbb{P}(\text{exactly 3 heads}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{(n-k)}.$$

8. Show that for arbitrary events A and B, we have:[2]

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

[2] For all of the remaining questions, start with the value on the left-hand side and proceed by applying steps based on our definitions that it can be manipulated to be equivalent to the right-hand side.

Solution: Note that $A \cup B$ may be written as $A \cup (B - A)$; since $A \cap (B - A) = \emptyset$, the third axiom implies that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B - A).$$

Similarly, since $B = (A \cap B) \cup (B - A)$, and $(A \cap B) \cap (B - A) = \emptyset$, we also have that

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(B - A).$$

We may rewrite this as

$$\mathbb{P}(B - A) = \mathbb{P}(B) - \mathbb{P}(A \cup B).$$

Substituting into the first equation yields the desired result

9. Show that if $A \subseteq B$, then $\mathbb{P}(B) \geq \mathbb{P}(A)$.

Solution: Since $A \subseteq B$, we may write

$$B = A \cup (B - A)$$

Moreover, the events A and $B - A$ are mutually exclusive. Thus, we have,

$$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B - A).$$

Since, by the first axiom, $\mathbb{P}(B - A) \geq 0$, we see that $\mathbb{P}(B) \geq \mathbb{P}(A)$, as claimed.

10. ($\star$) Show that if A and B are independent, then $\mathbb{P}(A|B^c) = \mathbb{P}(A)$.

Solution: This is only tricky because you have to apply just the right transformations in each step:

$$\begin{aligned}
\mathbb{P}(A|B^c) &= \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(B^c)} && \text{(def. of conditional prob.)} \\
&= \frac{\mathbb{P}(A) - \mathbb{P}(A \cap B)}{\mathbb{P}(B^c)} && \text{(Q4b on WS05)} \\
&= \frac{\mathbb{P}(A) - \mathbb{P}(A|B) \cdot \mathbb{P}B}{\mathbb{P}(B^c)} && \text{(def. of conditional prob.)} \\
&= \frac{\mathbb{P}(A) - \mathbb{P}(A|B) \cdot \mathbb{P}B}{1 - \mathbb{P}(B)} && \text{(Q4a on WS05)} \\
&= \frac{\mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}B}{1 - \mathbb{P}(B)} && \text{(independence)} \\
&= \mathbb{P}(A) \times \frac{1 - \mathbb{P}B}{1 - \mathbb{P}(B)} && \text{(factor common term)} \\
&= \mathbb{P}(A) && \text{(cancel)}
\end{aligned}$$