Worksheet 11 (Solutions)

1. We say that the random variable Y has a **Bernoulli distribution** with parameter p if it has the following pmf:

$$p_Y(y) = \begin{cases} (1-p), & \text{if } y = 0 \\ p, & \text{if } y = 1 \end{cases}$$

For some $p \in [0, 1]$. We can indicate this using the shorthand $Y \sim Bernoulli(p)$. What is $Var(Y)$?

Solution: Using the formula form the handout makes this easy because $Y = Y^2$ for all values of Y (either 0 or 1). Therefore:

$$\begin{aligned} Var(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \\ &= p - p^2 = (1-p) \cdot p \end{aligned}$$

2. Let $X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$, by which I mean that this is a set of independent random variables that all have a Bernoulli distribution with some fixed value of p.[1] If $Y = \sum_{i=1}^{n} X_i$ we say that Y has a **Binomial distribution** with parameters n and p. We can write this $Y \sim Bin(n, p)$. What are: (a) $p_Y(y)$, (b) $\mathbb{E}Y$, and (c) $Var(Y)$. Hint: We have already more-or-less done the first one and the second two can be done in an easy way.

[1] The notation i.i.d. stands for independent and identically distributed.

Solution: (a) Is exactly the same as question 7 on worksheet 7 and the pmf is given by:

$$p_Y(y) = \binom{n}{k} \cdot p^k \cdot (1-p)^{(n-k)}.$$

(b) If we work off of the definition in the question, we see that:

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} \mathbb{E}[X_i] \\ &= \sum_{i=1}^{n} p \\ &= np \end{aligned}$$

And (c) comes from the same derivation, keeping in mind that the X_i

variables are independent.

$$\begin{aligned} Var[Y] &= Var\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} Var[X_i] \\ &= \sum_{i=1}^{n} p(1-p) \\ &= np(1-p) \end{aligned}$$

3. Let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$ for an infinite sequence of random variables X_j. Define Y to be a random variable equal to the first value of j such $X_j = 1$. Then, we say that Y has a **geometric distribution** with parameter p and write $Y \sim Geom(p)$. What is $p_Y(y)$?

Solution: In order for the pmf to be exactly equal to y, we need to have had a sequence of $y-1$ 0s followed by a 1. Since the X_i are independent this just becomes:

$$p_Y(y) = (1-p)^{y-1} \times p$$

4. Let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$ for an infinite sequence of random variables X_j. Define Y to be a random variable equal to the number of X_j's required before we have k 1's. This is called the **negative binomial** distribution with parameters n and k. We write $Y \sim NB(n,k)$. What is $p_Y(y)$?[2]

[2] Don't worry, combinatorics questions are only coming back very briefly here.

Solution: In order for the pmf to be exactly equal to y, we need to have had a sequence of $y-1$ 0s followed by a 1. Since the X_i are independent this is just:

$$p_Y(y) = \binom{y-1}{k-1} \cdot (1-p)^{y-k} \times p^k.$$

5. The final common discrete random variable that we will need this semester is called the **Poisson distribution**. We will motivate where it comes from next time, but here is the pmf of a random variable Y with a Poisson distribution having parameter λ is:[3]

[3] The Taylor series of e^x around zero is $\sum_i x^n/n!$, which can be used to show that the pmf sums to 1.

$$p_Y(y) = \frac{\lambda^y e^{-\lambda}}{y!}.$$

And we write $Y \sim Poisson(\lambda)$. What is $\mathbb{E}Y$? Hint: Try to simplify the result so that it looks like the pmf, which we know sums to zero. Also, be careful about dividing by zero.

Solution: To start, note that we have the following:

$$\mathbb{E}Y = \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \cdot y$$

Notice that the first term multiplies by zero, so we can remove it from the sum. Then, this allows us to simplify the factorial (since $y/y! = 1/(y-1)!$):

$$\begin{aligned}\mathbb{E}Y &= \sum_{y=1}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \cdot y \\ &= \sum_{y=1}^{\infty} \frac{\lambda^y e^{-y}}{(y-1)!}\end{aligned}$$

Now, do a change of variables where $y = x + 1$:

$$\begin{aligned}\mathbb{E}Y &= \sum_{y=1}^{\infty} \frac{\lambda^y e^{-\lambda}}{(y-1)!} \\ &= \sum_{x=0}^{\infty} \frac{\lambda^{x+1} e^{-\lambda}}{(x)!} \\ &= \lambda \cdot \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x)!} \\ &= \lambda\end{aligned}$$

The last step comes because the second term is just the sum of the pmf of a Poisson random variable, which by definition sums to zero. This is a common trick that will be particularly important when we get to continuous random variables.

6. Go to the course website and click on the link at the top called "viz". It sends you to a page the visualizes the probability mass functions of common distributions. Take a few moments to see how the shapes of the five distributions defined here change as their parameters change. Note: The negative binomial is defined a little differently than in our notes in the visualization, but the idea is the same.

Solution: Please let me know if you have trouble accessing the visualizations.