Worksheet 14 (Solutions)

1. Let X be a continuous random variable defined over the set $[0, b]$ for some $b > 0$ whose density function is some constant value C over that interval and 0 otherwise. Find the constant C that makes this a valid density function.

Solution: We need the integral of the density to sum to 1, so we get:

$$\begin{aligned} 1 &= \int_{x=0}^{b} C dx \\ &= C \cdot [x]_{x=0}^{b} \\ &= C \cdot [b - 0]_{x=0}^{b} \\ &= C \cdot b \end{aligned}$$

So $C = \frac{1}{b}$. You likely could have figured that out visually as well, given that the integral is just the size of a rectangle with width b and height C.

2. What are $\mathbb{E}X$ and $Var(X)$ for X as defined in question 1?

Solution: The expected value is given by:

$$\begin{aligned} \mathbb{E}X &= \frac{1}{b} \cdot \int_{x=0}^{b} x dx \\ &= \frac{1}{b} \cdot \left[0.5 \cdot x^2\right]_{x=0}^{b} \\ &= \frac{1}{b} \cdot 0.5 \cdot b^2 = \frac{b}{2} \end{aligned}$$

The expected squared is given by:

$$\begin{aligned} \mathbb{E}X &= \frac{1}{b} \cdot \int_{x=0}^{b} x^2 dx \\ &= \frac{1}{b} \cdot \left[\frac{1}{3} \cdot x^3\right]_{x=0}^{b} \\ &= \frac{1}{3b} \cdot b^3 = \frac{b^2}{3} \end{aligned}$$

And so the variance is:

$$\begin{aligned} Var(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 \\ &= \frac{b^2}{3} - \frac{b^2}{2^2} \\ &= \frac{b^2}{12}. \end{aligned}$$

3. Let X be a continuous random variable with density $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and some fixed $\lambda > 0$. This is called the exponential distribution, which we can write $X \sim Exp(\lambda)$. What is the cumulative distribution $F(x)$? Find $\mathbb{P}[x \geq 1]$.

Solution: The cumulative distribution is given by the following integral, where we use the substitution $u = \lambda \cdot x \rightarrow du = \lambda dx$:

$$\begin{aligned} F(z) &= \int_0^z f(x)dx \\ &= \int_0^z \lambda \cdot e^{-\lambda \cdot x} dx \\ &= \lambda \cdot \int_0^z e^{-\lambda \cdot x} dx \\ &= \lambda \cdot \frac{1}{\lambda} \cdot \int_0^{\lambda z} e^{-u} du \\ &= \left[-e^{-u}\right]_{x=0}^{\lambda z} \\ &= 1 - e^{-\lambda z} \end{aligned}$$

The desired probability comes from the CDF:[1]

$$\begin{aligned} \mathbb{P}[x \geq 1] &= 1 - \mathbb{P}[x < 1] \\ &= 1 - [1 - e^{-\lambda}] \\ &= e^{-\lambda}. \end{aligned}$$

[1] With continuous random variables, we do not need to be careful about the difference between $\geq$ and $>$ because the probability that the variable will take on an exact value at the endpoint is zero.

4. Let $X \sim Exp(\lambda)$ and fix two constants $a > 0$ and $b > 0$. We can define the following:

$$\mathbb{P}[X > a + b | X > a] = \frac{\mathbb{P}[(X > a + b) \cap (X > a)]}{\mathbb{P}[(X > a)]} = \frac{\mathbb{P}[X > a + b]}{\mathbb{P}[(X > a)]}$$

Calculate this quantity using the CDF of the exponential, and try to simplify the result in terms of a probability. The exponential is used to model wait times between independent events, largely due to the property you should see here.

Solution: When know that $\mathbb{P}[X > z]$ is equal to $e^{-\lambda z}$ for any positive value of z from the previous question. So, this is actually fairly straightforward:

$$\begin{aligned} \mathbb{P}[X > a + b | X > a] &= \frac{\mathbb{P}[X > a + b]}{\mathbb{P}[(X > a)]} \\ &= \frac{e^{-\lambda(a+b)}}{e^{-\lambda(a)}} = e^{-\lambda(a+b-a)} = e^{-\lambda b} \end{aligned}$$

And, what is this quantity? It's just the probability that X is greater than b. So, we have:

$$\mathbb{P}[X > a + b | X > a] = \mathbb{P}[X > b]$$

This is called the *memorylessness* property. Conceptually, if use it to model wait times, it says that the wait time for the next event is independent of the most recent previous event occured.

5. Find the MGF of the exponential distribution for $t < \lambda$.

Solution: The MFG is given by:

$$\begin{aligned} m_X(t) &= \mathbb{E}e^{tX} \\ &= \lambda \cdot \int_0^\infty e^{tx} \cdot e^{-\lambda x} dx \\ &= \lambda \cdot \int_0^\infty e^{-(\lambda - t)x} dx \\ &= \lambda \cdot \left[\frac{1}{\lambda - t} \cdot e^{(\lambda - t)x} \right]_{x=0}^\infty \\ &= \frac{\lambda}{\lambda - t} \cdot \left[1 - e^{(\lambda - t)\cdot\infty} \right] \\ &= \frac{\lambda}{\lambda - t}. \end{aligned}$$

Where in the last step we used the fact that $t < \lambda$ to argue that $e^{-(\lambda - t)x}$ should limit to 0 as $x \to \infty$

6. If $X \sim Exp(\lambda)$, find $\mathbb{E}X$ and $Var(X)$.

Solution: These come from the moment generating function. The first two derivatives are:

$$\begin{aligned} \frac{\partial}{\partial t} m_X(t) &= \frac{\partial}{\partial t} \left(\frac{\lambda}{\lambda - t} \right) \\ &= \lambda \cdot (-1) \cdot (\lambda - t)^{-2} \cdot (-1) \\ &= \lambda \cdot (\lambda - t)^{-2} \end{aligned}$$

And,

$$\begin{aligned} \frac{\partial^2}{\partial^2 t} m_X(t) &= \frac{\partial}{\partial t} \left(\lambda \cdot (\lambda - t)^{-2} \right) \\ &= \lambda \cdot (-2) \cdot (\lambda - t)^{-3} \cdot (-1) \\ &= 2\lambda \cdot (\lambda - t)^{-3} \end{aligned}$$

Evaluating at $t = 0$ we get:

$$\begin{aligned} \mathbb{E}X &= \lambda^{-1} \\ \mathbb{E}X^2 &= 2\lambda^{-2} \end{aligned}$$

And, finally, the variance formula yields,

$$\begin{aligned} Var(X) &= 2\lambda^{-2} - \left(\lambda^{-1} \right)^2 \\ &= \lambda^{-2}. \end{aligned}$$

7. Consider working as a PA in an urgent care facility late at night. On average, you know that a patient comes in every 10 minutes. You need to go to the back room to restock the insulin in your front office, which takes 3 minutes. Using the exponential distribution, how likely is it that a patient will arrive while you are gone?

Solution: We need to model the time inbetween patients X with an exponential that has a mean of 10, so $\lambda = 0.1$. Then, we just need the following:

$$\mathbb{P}[X < 3] = 1 - e^{-\lambda \cdot 3} = 1 - e^{-0.1 \cdot 3} = 1 - e^{-0.3} \approx 0.259$$

So, there is about a 1/4 chance of missing the arrival of a patient.

8. If the wait times between events is distributed as $Exp(\lambda)$ then the number of events that occurs in any interval of size t is given by a random variable that has a Poisson distribution with rate $t \cdot \lambda$. Using the data from the previous example, how many patients do you see on average over an 8 hour shift?

Solution: Remember that the model from before is in minutes, so your shift is 480 minutes. The mean of a Poisson is just equal to its rate parameter, so the average number of patients is $480 \cdot 0.1 = 48$.