Worksheet 16 (Solutions)

1. Let $X \sim Gamma(\alpha, \beta)$. Find $\mathbb{E}X$ using the moment generating function.

Solution: The first derivative of the mgf is given by:

$$\begin{aligned}\frac{d}{dt}\left[m_X(t)\right] &= \frac{d}{dt}\left[(1-\beta t)^{-\alpha}\right] \\ &= -\alpha \cdot (1-\beta t)^{-\alpha-1} \cdot (-\beta) \\ &= \alpha \cdot \beta \cdot (1-\beta t)^{-\alpha-1}\end{aligned}$$

Setting $t = 0$ gives the expected value to be:

$$\begin{aligned}\mathbb{E}X &= \alpha \cdot \beta \cdot (1-\beta(0))^{-\alpha-1} \\ &= \alpha\beta\end{aligned}$$

Which is what we have on our distribution sheet as well.

2. Let $X \sim Gamma(\alpha, \beta)$. Find $Var(X)$ using the moment generating function.

Solution: Starting with the first derivative derived in the previous question, the second derivative of the mgf is:

$$\begin{aligned}\frac{d^2}{dt^2}\left[m_X(t)\right] &= \frac{d}{dt}\left[\alpha \cdot \beta \cdot (1-\beta t)^{-\alpha-1}\right] \\ &= \alpha \cdot \beta \cdot (-\alpha-1) \cdot (1-\beta t)^{-\alpha-2} \cdot (-\beta) \\ &= \alpha \cdot \beta^2 \cdot (\alpha+1) \cdot (1-\beta t)^{-\alpha-2}\end{aligned}$$

Setting $t = 0$ gives the second moment to be:

$$\begin{aligned}\mathbb{E}X^2 &= \alpha \cdot \beta^2 \cdot (\alpha+1) \cdot (1-\beta t)^{-\alpha-2} \\ &= \alpha \cdot \beta^2 \cdot (\alpha+1)\end{aligned}$$

And the variance as:

$$\begin{aligned}Var(X) &= \mathbb{E}X^2 - \left[\mathbb{E}X\right]^2 \\ &= \alpha \cdot \beta^2 \cdot (\alpha+1) - \alpha^2\beta^2 \\ &= \alpha^2 \cdot \beta^2 + \alpha\beta^2 - \alpha^2\beta^2 \\ &= \alpha\beta^2\end{aligned}$$

Which also matches the value on the reference sheet.

3. Let $X \sim Gamma(\alpha, \beta)$. Show that $c \cdot X$ also has a Gamma distribution and find it's parameters. Hint: Use the moment generating function.

Solution: Notice that we can manipulate the mgf to move the constant c from the variable X to the argument t:

$$m_{cX}(t) = \mathbb{E}e^{t(cX)} = \mathbb{E}e^{(tc)X} = m_X(ct)$$

So, we then have:

$$\begin{aligned}
m_{cX}(t) &= m_X(ct) \\
&= (1 - \beta ct)^{-\alpha} \\
&= (1 - (\beta c) \cdot t)^{-\alpha}
\end{aligned}$$

And we see that this is the mgf of a random variable with a $Gamma(\alpha, \beta c)$ distribution.

4. Let $X \sim Gamma(\alpha, \beta)$ and $Y \sim Gamma(\theta, \beta)$ be independent random variables. Find the distribution of $Z = X + Y$.

Solution: The mgf of a sum of independent random variable is the product of the the mgf functions of the two parts. So:

$$\begin{aligned}
m_Z(t) &= m_X(t) \cdot m_Y(t) \\
&= (1 - \beta t)^{-\alpha} \cdot (1 - \beta t)^{-\theta} \\
&= (1 - \beta t)^{-(\alpha+\theta)}
\end{aligned}$$

And therefore $Z \sim Gamma(\alpha + \theta, \beta)$.

5. Let $X \sim Gamma(\alpha, \beta)$. For a sufficently large α, (a) how and (b) why can we approximate X by a normal distribution?

Solution: From the previous question we can see that, for example, if α is an integer, X is the sum of α independent random variables that have a distribution of $Gamma(1, \beta)$. Since this is a sum of independent random variables, we can approximate it with a normal distribution. From the first two questions we know that this will have an expected value of $\alpha\beta$ and a variance of $\alpha\beta^2$, so $X \dot{\sim} N(\alpha\beta, \alpha\beta^2)$.

6. The normalizing constant in the Beta distribution gives that the following must true for any positive α and β:

$$\left[\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right] = \int_0^1 \left[x^{\alpha-1}(1-x)^{\beta-1}\right] dx$$

Let $X \sim Beta(\alpha, \beta)$. Find $\mathbb{E}(X)$. Start by writing down anx integral definition of the expected value, but use the formula above to solve it. Note that you can simplify the final result without using the Gamma function.

Solution: Using the integral form of the expected value, we get the following:

$$\begin{aligned}\mathbb{E}X &= \int_0^1 \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[x^{\alpha-1}(1-x)^{\beta-1}\right] x\,dx \\ &= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \int_0^1 \left[x^{\alpha}(1-x)^{\beta-1}\right] dx\end{aligned}$$

The integrand is the formula from the question with $\alpha \to \alpha+1$, so we have:

$$\mathbb{E}X = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\right]$$

And, now we can use the rule that $\Gamma(z+1) = z\Gamma(z)$ to simplify the terms with a $+1$:

$$\begin{aligned}\mathbb{E}X &= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[\frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}\right] \\ &= \frac{\alpha}{\alpha+\beta}\end{aligned}$$

And this matches what is on the reference table.

7. Let $X \sim Beta(\alpha,\beta)$. Using the same trick as the previous question, what is $\mathbb{E}X^2$?

Solution: The value of $\mathbb{E}X^2$ will just have an $\alpha+1$ where the previous question had an α (and the density has an $\alpha-1$). If you want to see it worked out, we have:

$$\begin{aligned}\mathbb{E}X^2 &= \int_0^1 \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[x^{\alpha-1}(1-x)^{\beta-1}\right] x^2\,dx \\ &= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \int_0^1 \left[x^{\alpha+1}(1-x)^{\beta-1}\right] dx\end{aligned}$$

Now, we have our original formula, but with $\alpha+2$ in place of α, which gives:

$$\mathbb{E}X^2 = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[\frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}\right]$$

Applying the rule that $\Gamma(z+1) = z\Gamma(z)$ twice to simplify the terms with a $+2$, we have:

$$\begin{aligned}\mathbb{E}X &= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[\frac{(\alpha+1)(\alpha)\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)}\right] \\ &= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}\end{aligned}$$

This does not match the value on the reference sheet, because it is just the second moment, not the variance. You could get the variance using out normal formula, but I didn't ask you to do that because it's just some messy algebra to get these all into a common fraction.