Worksheet 18 (Solutions)

1. The uniform distrbution $U(a,b)$ has a constant pdf equal to $(b-a)^{-1}$ between $a<b$ and equal to 0 otherwise. Let $U \sim U(0,1)$ and define $Y = U^2$. Find the PDF of Y and determine what distribution (it is one that we have studies already) it comes from. Hint: Remember to write the final equation in terms of y.

Solution: We start by determining the derivative

$$\begin{aligned}\left|\frac{d}{dy}g^{-1}(y)\right| &= \left|\frac{d}{dy}\sqrt{y}\right| \\ &= \left|\frac{1}{2}y^{-1/2}\right|\end{aligned}$$

And then:

$$\begin{aligned}f_Y(y) &= f_U(g^{-1}(y)) \cdot \left|\frac{1}{2}y^{-1/2}\right| \\ &= \frac{1}{2}y^{-1/2}\end{aligned}$$

This is just the Beta distribution with $\alpha = 1/2$ and $\beta = 1$.

2. Let $U \sim U(0,1)$ and define $Y = U^{1/2}$. Find the PDF of Y and determine what distribution (it is one that we have studies already) it comes from.

Solution: This is very similar to the last question. We start by determining the derivative

$$\begin{aligned}\left|\frac{d}{dy}g^{-1}(y)\right| &= \left|\frac{d}{dy}y^2\right| \\ &= |2y|\end{aligned}$$

And then:

$$\begin{aligned}f_Y(y) &= f_U(g^{-1}(y)) \cdot |2y| \\ &= 2y\end{aligned}$$

This is just the Beta distribution with $\alpha = 2$ and $\beta = 1$. It's not much more difficult to prove from this that in general $U^{\frac{1}{\alpha}}$ has a Beta(α, 1) distribution.

3. Let $Z \sim N(0,1)$ and consider the random variable $Y \sim Z^2$. We cannot directly apply the change of variables formula because $g(z) = z^2$ is not a one-to-one function (it maps positive numbers to the same number as a negative number). We can fix this by considering a random variable $X = |Z|$ and then defining Y to (equivalently) be equal to X^2. The density of X is just twice the density of a standard normal, but only for positive values of x:

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-x^2/2}, \quad x > 0.$$

Use the change of variables formula to derive the density of Y, which we will call χ_1^2 as on the handout.

Solution: The function g is the same as in question 1, so we have:

$$\begin{aligned} \left| \frac{d}{dy} g^{-1}(y) \right| &= \left| \frac{d}{dy} \sqrt{y} \right| \\ &= \left| \frac{1}{2} y^{-1/2} \right| \end{aligned}$$

Note that this will always be positive for $y > 0$. Then, the density is given by:

$$f_Y(y) = \frac{\sqrt{2}}{\sqrt{pi}} e^{-y/2} \cdot \left[\frac{1}{2} y^{-1/2} \right], \quad y > 0$$

We will simplify this in the next question.

4. The value of $\Gamma(1/2)$ is equal to $\sqrt{\pi}$. Use this fact to manipulate the density you have in the previous question, which we called χ_1^2, is also a form of the Gamma distribution.

Solution: The power in the exponent and the value in the Gamma functions hints that this must be Gamma(1/2, 2) distribution. We can see that it does re-arrange to the correct form:

$$f_Y(y) = \frac{1}{\Gamma(1/2) 2^{1/2}} y^{(1/2)-1} e^{-y/2}, \quad y > 0$$

As desired.

5. Let $Z_1, \ldots, Z_n \overset{\text{i.i.d.}}{\sim} N(0,1)$. If we have $Y = \sum_i Z_i^2$, then we say that Y follows a chi-squared distribution with k degrees of freedom. We write this as $Y \sim \chi_k^2$. Using the results from the previous two questions, (a) what is another name for this distribution? (b) What are the mean, variance, and mfg of Y? Hint: The second part should be easy.

Solution: (a) We know that χ_1^2 is just Gamma(1/2, 2). From the definition, a random variable distributed as χ_k^2 can be constructed by taking the sum of k independent random variables distributed as χ_1^2. Finally, we know that the sum of independent Gamma distributions with the same second parameter is another Gamma with the sum of the first parameter. Therefore, $\chi_1^2 \equiv Gamma(k/2, 2)$.

(b) This just comes right off the reference sheet (not the one about the χ_k^2, that's cheating) for the Gamma distribution. The mean is $k/2{\cdot}2 = k$, the variance is $k/2 \cdot 2^2 = 2k$, and the mfg is $(1-2t)^{-(k/2)}$.

6. Let $U \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$ be a random variable. Define $T = tan(U)$. Use the change of variable formula to determine the form of the pdf of T.[1] We have not seen this distribution before. It is called the (standard) Cauchy distribution, and is included on the reference sheet. It is a very interesting distribution because it has no defined mean or variance.

[1] The derivative of $tan^{-1}(t)$ is $1/(1+t^2)$.

Solution: Let's try to find the PDF of T. The derivative is given by:

$$\frac{du}{dt} = \frac{d}{dt}\left[tan^{-1}(t)\right] = \frac{1}{t^2+1}$$

Note that the density of U is $\frac{1}{\pi}$ for all values of u between $\pm\pi/2$. This gives then that:

$$\begin{aligned} f_T(y) &= f_U(u) \cdot |det(J_{g^{-1}})| \\ &= \frac{1}{\pi} \cdot \frac{1}{t^2+1} \\ &= \frac{1}{\pi(t^2+1)} \end{aligned}$$

This example presents an interesting justification of the Cauchy distribution. Stand in front of a designated point on an infinitely long wall holding a flashlight. Pick an angle uniformily somewhere between your immediate left and your immediate right and shine the flashlight at the wall from this angle. The location of the light on the wall is given by the tangent of the angle you choose relative to the wall. Therefore, the location on the wall will follow a Cauchy distribution. While this setup may sound quite strange, it is a great mental image to help understand some of the seemingly odd behaviors of the distribution.

7. There is also a two-dimensional change of variables formula. It's not difficult to write-out, but solving it can get messy. It can be used to derive, for example, for independent $U_1 \sim \chi^2_{k_1}$ and $U_2 \sim \chi^2_{k_2}$ the distribution of $F = \frac{U_1/k_1}{U_2/k_2}$. This is called the F-distribution. Or, for an independent $Z \sim N(0,1)$ and $U \sim \chi^2_k$, the distribution of $T = \frac{Z}{\sqrt{U/k}}$. This is called the Student-T distribution. These are both important distrbutions in statistics, but the derivations are quite messy. What is an adjective describing how happy you are not to have to derive them?

Solution: Thesaurus.com suggests the following possibilities: cheerful, contented, delighted, ecstatic, elated, glad, joyful, joyous, jubilant, lively, merry, overjoyed, peaceful, pleasant, pleased, satisfied, thrilled, upbeat, blessed, blest, blissful, blithe, can't complain, captivated, chipper, chirpy, content, convivial, exultant, flying high, gay, gleeful, gratified, intoxicated, jolly, laughing, light, looking good, mirthful , on cloud nine, peppy, perky, playful, sparkling, sunny, tickled, tickled pink, up, walking on air. Other answers are also possible.