Worksheet 19

1. Let $Z \sim N(0,1)$. It can be shown that $\mathbb{E}|Z| = \sqrt{2/\pi}$. Use Markov's inequality to bound the probabilities: (a) $\mathbb{P}[|Z| > 1.28]$, (b) $\mathbb{P}[|Z| > 1.96]$, (c) $\mathbb{P}[|Z| > 2.58]$, and (d) $\mathbb{P}[|Z| > 3.89]$. Compare these to the exact quantities on the handout.

2. Chebychev's inequality (see the reference sheet) can be derived directly from Markov's inequality. Let X be a random variable with mean μ and variance σ^2. Define $Y = (X - \mathbb{E}X)^2$ and apply Markov's inequality with $X \to Y$ and $a \to a^2$ (remember, a can be any positive constant so we can replace it with a squared version of itself if we do so on both sides). Plug the value of Y back in, use the definition of variance, and simplify to derive Chebychev's inequality

3. Let $Z \sim N(0,1)$. Use Chebychev's inequality to bound the probabilities: (a) $\mathbb{P}[|Z| > 1.28]$, (b) $\mathbb{P}[|Z| > 1.96]$, (c) $\mathbb{P}[|Z| > 2.58]$, (d) $\mathbb{P}[|Z| > 3.89]$. Compare these to the previous results. Which ones are tighter?

4. Chernoff's inequality (see the reference sheet) can also be derived directly from Markov's inequality. Let X be a random variable with a well-defined moment generating function. Apply Markov's inequality with $|X| \to e^{tX}$ (the new value is also positive, so no need for absolute value) and $a \to e^{ta}$. Simplify the part inside of the probability on the left-hand side to derive Chernoff's inequality.

5. Chernoff's inequality has an extra term in it, the t, that provides a whole family of bounds for a given value of a. The tightest bound depends on the distribution. Let $Z \sim N(0,1)$. Using the moment generating function, what value of t provides the tightest bound on $\mathbb{E}[Z \geq a]$?

6. Let $Z \sim N(0,1)$. Use Chernoff's inequality (and the tightest value of t from the previous question) to compute bounds on the following: (a) $\mathbb{P}[|Z| > 1.28]$, (b) $\mathbb{P}[|Z| > 1.96]$, (c) $\mathbb{P}[|Z| > 2.58]$, and (d) $\mathbb{P}[|Z| > 3.89]$. Note that due to the symmetry of the normal distribution, you can double the probability that Z is larger than some a to get the probability that $|Z|$ is larger than a. You should notice an interesting pattern relative to the other bounds that we have.

7. (Weak Law of Large Numbers) Let's finish with a result that shows the power of these tail inequalities for establishing theoretical results. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables that come from a distribution with finite mean μ and finite variance σ^2. For any positive

n, define the sample mean to be:

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$$

Then, for any $\epsilon > 0$:

$$\lim_{n \to \infty} \mathbb{P}(|\bar{X}_n - \mu| > \epsilon) = 0.$$

Prove that this is true using Chebyshev's inequality. Hint: Compute the mean and variance of $\bar{X}_n$ and then just apply the theorem as-is.