

Handout 06: Hypothesis Testing

An hypothesis test is a way of establishing whether a random sample offers support for a particular conclusion about the population distribution \mathcal{G} . The mechanics of hypothesis testing are similar to those of confidence intervals, but the interpretation and terminology differs. For an hypothesis test, we start with a **null hypothesis** H_0 and see if the data support rejecting this baseline assumption in favor of an **alternative hypothesis** H_A .

In its formal specification, to do an hypothesis test we first select a **significance level** $(1 - \alpha)$ and construct an event R , called a **rejection region**, such that $\mathbb{P}[R|H_0] \leq \alpha$. If we then observe some data and R occurs, we **reject** the null hypothesis in favor of the alternative. Otherwise, we **retain** (or fail to reject) the null hypothesis. The goal is to create a rejection region that has a much higher probability of occurring under the alternative hypothesis than it does under the null.

The standard approach to creating a rejection region is to start with a pivot, plug in the parameters of the null distribution, and then form a **test statistic** with a known distribution under H_0 . Then, we can create something similar to a confidence interval around the test statistic and use the complement of that as the rejection region. Let's see an example! Assume that you have a random sample from a normal distribution with unknown mean and unknown variance. We will do an hypothesis test with $H_0 : \mu = 1$ and $H_A : \mu \neq 1$. We form a test statistic T using the pivot quantity that we saw last week, but with μ_X replaced with the null hypothesis value of 1:¹

$$T = \frac{\bar{X} - 1}{\sqrt{S_X^2/n}}$$

Under H_0 , $T \sim t(n - 1)$. A rejection region R can be given by the following event:

$$R = \{-t_{\alpha/2} \leq T \leq t_{\alpha/2}\}^c.$$

Due to the way it is defined, this will have the desired property that $\mathbb{P}[R|H_0] \leq \alpha$. A common shorthand for the rejection region is to call the quantity $t_{\alpha/2}$ a **critical value**, and then simply reject the null hypothesis if the (absolute value) of the test statistic exceeds the critical value. Hypothesis test procedures are often named based on the distribution of the test statistic.² Here, we have derived what is called the **one-sample T-test**.

There is another related way of doing hypothesis testing that, at least initially, forgoes the language about rejecting or retaining the null hy-

¹ I have reversed the numerator from the previous notes to follow the standard convention that a positive T corresponds to a mean higher than the null hypothesis. Due to the symmetry of the t-distribution, T still has the same t-distribution either way.

² This is not an ideal convention because many different tests can have the same test statistic distribution. The naming convention causes constant confusion. We can talk more about ways to mitigate these issues in your own work.

pothesis. Instead of having a fixed critical value, we could ask the question: what is the smallest value of α such that corresponding test statistic would be equal to (or greater than) the critical value? For example, if we observe a positive value of t from a T-test, we could define p such that:

$$\mathbb{P}[t \leq T] = p.$$

This quantity is called a **p-value**. A p-value is usually the form that statistical software will report the results of an hypothesis test and is increasingly the way that results are communicated through. Note that we can recover the concept of statistical significance by simply checking the p-value is less than the significant level. If it is, we have a statistically significant result and would reject the null hypothesis in favor of the alternative hypothesis.

One can always use the p-value as a number that we use to compare to our specific values of α and then determine whether to reject or retain H_0 . There is a lot of discussion in statistics about how the p-value itself should be interpreted beyond this, if at all.