Handout 10: Likelihood-Ratio Test

In addition to the nice point-estimator properties of MLE estimators, they also come along with a built-in general procedure for building hypothesis tests. This process is called the **likelihood-ratio test**; while it only generates asymptotically correct tests in the limit of large data, it often performs very well even for small datasets.

It will be helpful for us to a start with a slightly more general description of the test than we needed for the introductory methods we covered in the first weeks. Namely, we will extend the idea of a simple hypothesis, where H_0 is a single value of the parameters, to complex hypotheses where H_0 is a general subset of all possibilities. Note that all of the elements we have already described, such p-values and rejection regions, still apply. Those concepts just need to hold for any value in the null hypothesis. Assume that we are working with a distribution that has k unknown parameters. We are going to define a null-hypothesis as a subset of all possible configurations of the unknown parameters, which is in turn a subset of $\mathbb{R}^k$. Symbolically, we have $\Theta_0 \subset \Theta \subseteq \mathbb{R}^k$.

Given a null hypothesis that $\theta \in \Theta_0$, we can define the quantity Λ as the following:

$$\Lambda = -2 \cdot \log \left[\frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta)}{\sup_{\theta \in \Theta} \mathcal{L}(\theta)} \right].$$

The value inside of the brackets must be between zero and one, and therefore Λ will always be positive. If we define θ_0 as the argmax of the numerator and $\hat{\theta}$ as the argmax of the numerator, this can be simplified in terms of the log-likelihood $l(\cdot)$:

$$\Lambda = 2 \cdot \left[l(\hat{\theta}) - l(\theta_0) \right].$$

There is an important (but not easy to prove) result called **Wilks' Theorem** that establishes that Λ will be approximately distributed as a χ^2. The degrees of freedom will be the difference in dimensionality between Θ and Θ_0. It is therefore an asymptotic pivot quantity.