Worksheet 18 (Solutions)

1. Consider a prior distribution where $\lambda \sim Gamma(\alpha, \beta)$ and a likelihood $X_j|\lambda \sim Poisson(\lambda)$ for an i.i.d. sample of size n. Find (a) the posterior distribution, (b) the Bayesian point estimator, and (c) the limit of the point estimator when the data dominates the prior.

Solution: For (a), we have:

$$\begin{aligned}
f_{\lambda|\vec{x}}(\lambda|\vec{x}) &\propto \left[\prod_j f_{x_j|\lambda}(x_j)\right] \times f_\lambda \\
&\propto \left[\prod_j \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}\right] \times \left[\frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \lambda^{\alpha-1} e^{-\lambda/\beta}\right] \\
&\propto \lambda^{\sum_i x_i} \cdot e^{-n\lambda} \cdot \lambda^{\alpha-1} \cdot e^{-\lambda/\beta} \\
&\propto \lambda^{\alpha+\sum_i x_i - 1} \cdot e^{-\lambda[\beta^{-1}+n]} \\
&\sim Gamma(\alpha + \sum_i x_i, [\beta^{-1}+n]^{-1}).
\end{aligned}$$

Now that we know the posterior distribution, the answer to (b) comes from the reference table:

$$\hat{\lambda}_{Bayes} = \mathbb{E}[\lambda|\vec{X}] = \frac{\alpha + \sum_i x_i}{\beta^{-1}+n}.$$

Finally, in the limit of large data, we see that (c) we have:

$$\hat{\lambda}_{Bayes} \to \frac{\sum_i x_i}{n} = \bar{x}.$$

So, in the limit of large data, the Bayes estimator limits to the MLE.

2. Assume you have a sample of size $n = 10$ from a Poisson distribution. The average of the data is $\bar{x} = 3$. What are the (a) MLE estimator of λ, (b) the Bayesian estimator of λ with a $Gamma(1,1)$, and (c) the Bayesian estimator of λ with a $Gamma(10,1)$?[1]

[1] Take a moment to compare the results and see how the relate the means of the two priors.

Solution: (a) The MLE is:

$$\hat{\lambda}_{MLE} = \bar{x} = 3.$$

(b) The Bayes estimator with this prior is:

$$\hat{\lambda}_{Bayes} = \frac{\alpha + \sum_i x_i}{\beta^{-1}+n} = \frac{1+30}{1+10} = \frac{31}{11} = 2.818$$

(c) The Bayes estimator with this prior is:

$$\hat{\lambda}_{Bayes} = \frac{\alpha + \sum_i x_i}{\beta^{-1}+n} = \frac{10+30}{1+10} = \frac{41}{11} = 3.727$$

3. Consider a prior distribution where $\lambda \sim Gamma(\alpha, \beta)$ and a likelihood $X|\lambda \sim Exp(\lambda)$ for an i.i.d. sample of size n.. Find (a) the posterior distribution, (b) the Bayesian point estimator, and (c) the limit of the point estimator when the data dominates the prior.

Solution: For (a), we have:

$$\begin{aligned} f_{\lambda|\vec{x}}(\lambda|\vec{x}) &\propto \left[\prod_j f_{x_j|\lambda}(x_j)\right] \times f_\lambda \\ &\propto \left[\prod_j \lambda e^{-\lambda x_i}\right] \times \left[\frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \lambda^{\alpha-1} e^{-\lambda/\beta}\right] \\ &\propto \lambda^n e^{-\lambda \sum_i x_i} \lambda^{\alpha-1} e^{-\lambda/\beta} \\ &\propto \lambda^{\alpha+n-1} e^{-\lambda[\sum_i x_i + \beta^{-1}]} \\ &\sim Gamma(\alpha + n, [\sum_i x_i + \beta^{-1}]^{-1}). \end{aligned}$$

The point estimator (b) comes from the table, just as in the first question:

$$\hat{\lambda}_{Bayes} = \mathbb{E}[\lambda|\vec{X}] = \frac{\alpha + n}{\sum_i x_i + \beta^{-1}}.$$

Finally, in the limit of large data, we see that (c) we have:

$$\hat{\lambda}_{Bayes} \to \frac{n}{\sum_i x_i} = \bar{x}^{-1}.$$

So, in the limit of large data, once again the Bayes estimator limits to the MLE.

4. Assume you have a sample of size $n = 30$ from an Exponential. The average of the data is $\bar{x} = 0.5$. What are the (a) MLE estimator of λ, (b) the Bayesian estimator of λ with a $Gamma(1, 1)$, and (c) the Bayesian estimator of λ with a $Gamma(1, 4)$?

Solution: (a) The MLE is:

$$\hat{\lambda}_{MLE} = \frac{1}{\bar{x}} = \frac{1}{0.5} = 2.$$

(b) The Bayes estimator with this prior is:

$$\hat{\lambda}_{Bayes} = \frac{\alpha + n}{\sum_i x_i + \beta^{-1}} = \frac{1 + 30}{15 + 1} = \frac{31}{16} = 1.938$$

(c) The Bayes estimator with this prior is:

$$\hat{\lambda}_{Bayes} = \frac{\alpha + n}{\sum_i x_i + \beta^{-1}} = \frac{1 + 30}{15 + 0.25} = \frac{31}{15.25} = 2.032$$

5. Consider a prior distribution where $p \sim Beta(a, b)$ and a likelihood $X|p \sim Geometric(1, \beta)$for an i.i.d. sample of size n. Find (a) the posterior distribution, (b) the Bayesian point estimator, and (c) the limit of the point estimator when the data dominates the prior.

Solution: For (a), we have:

$$\begin{aligned}
f_{p|\vec{x}}(p|\vec{x}) &\propto \left[\prod_j f_{x_j|p}(x_j)\right] \times f_p \\
&\propto \left[\prod_j (1-p)^{x_i-1} \cdot p\right] \times \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot p^{a-1}(1-p)^{b-1}\right] \\
&\propto (1-p)^{\sum_i x_i - n} \cdot p^n \cdot p^{a-1} \cdot (1-p)^{b-1} \\
&\propto (1-p)^{b+\sum_i x_i - n - 1} \cdot p^{a+n-1} \\
&\sim Beta(a+n, b+\sum_i x_i - n)
\end{aligned}$$

The point estimator (b) comes from the table, just as in the first question:

$$\hat{p}_{Bayes} = \mathbb{E}[p|\vec{X}] = \frac{a+n}{a+n+b+\sum_i x_i - n} = \frac{a+n}{a+b+\sum_i x_i}$$

Finally, in the limit of large data, we see that (c) we have:

$$\hat{p}_{Bayes} \to \frac{n}{\sum_i x_i} = \frac{1}{\bar{x}}.$$

So, in the limit of large data, the Bayes estimator limits to the MLE.

6. Assume you have a sample of size $n = 12$ from a Geometric distribution. The average of the data is $\bar{x} = 2$. What are the (a) MLE estimator of p, (b) the Bayesian estimator of λ with a $Beta(1, 10)$, and (c) the Bayesian estimator of λ with a $Beta(10, 1)$? What are the means of the two priors?

Solution: (a) The MLE is:

$$\hat{p}_{MLE} = \frac{1}{\bar{x}} = \frac{1}{2} = 0.5$$

(b) The Bayes estimator with this prior is:

$$\hat{p}_{Bayes} = \frac{a+n}{a+b+\sum_i x_i} = \frac{1+12}{1+10+24} = \frac{13}{35} = 0.371$$

(c) The Bayes estimator with this prior is:

$$\hat{p}_{Bayes} = \frac{a+n}{a+b+\sum_i x_i} = \frac{10+12}{10+1+24} = \frac{22}{35} = 0.629$$